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Unified Finite Element Method for Engineering Systems with Hybrid Uncertainties

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The finite element method (FEM) is established as one of the most popular and powerful tools in analysis and design of engineering systems. However, the method can only be used at an advanced (or detail) design stage when a designer has developed sufficient confidence in the design scheme. In the early (or conceptual/preliminary) design phase, the method cannot be used directly due to the existence of uncertainties of various types. A procedure is proposed for the FEM to handle engineering systems in the presence of hybrid uncertainties that are characterized by randomness (or stochastic uncertainty) and fuzziness (or design imprecision). The proposed scheme utilizes the techniques of stochastic and fuzzy FEMs in a unified manner to investigate the effects of both input random and fuzzy uncertainties on the output system response. Interpretations of the generated solution space and guidelines in selection of an appropriate solution, according to specific design requirements, are discussed. Two numerical examples are presented to illustrate the computational aspects of the proposed scheme. The current procedure is expected to enhance the understanding of designers regarding the effects of uncertainties in an evolutionary design process, especially in the conceptual or preliminary design phase.

Nomenclature

$\boldsymbol{A}$	= coefficient matrix in a linear system ($\boldsymbol{AX} = \boldsymbol{B}$)
$\boldsymbol{B}$	= vector of constants in a linear system ($\boldsymbol{AX} = \boldsymbol{B}$)
d_i	= ith design parameter
$E[\cdot]$	= expected value
e_i	= ith empirical parameter
$f(\cdot)$, $f_i(\cdot)$	= probability density function
$\boldsymbol{H}(\cdot)$	= linear system ($\boldsymbol{H} = \boldsymbol{AX}$)
$h_i(\cdot)$	= ith linear equation
$m(\cdot)$, $m_i(\cdot)$	= membership function
m_X	= hybrid-uncertainty mean
$\boldsymbol{P}$	= vector of input parameters
$\boldsymbol{P}_d$	= vector of design parameters
$\boldsymbol{P}_e$	= vector of empirical parameters
p_i	= ith input parameter
$\boldsymbol{X}$, $\boldsymbol{X}_i$	= vector of unknown variables
x_i	= ith unknown variable
α	= design level cut
$\Delta^L_{(\cdot)}$, $\Delta^R_{(\cdot)}$	= vector of left/right spreads
$\delta^L_{(\cdot)}$, $\delta^R_{(\cdot)}$	= left/right spread ($\geq$0)
$\mu(\cdot)$, $\mu_i(\cdot)$	= mean value
ν_X	= hybrid-uncertainty variance
$\sigma(\cdot)$, $\sigma_i(\cdot)$	= standard deviation ($\geq$0)
Ω	= universe of discourse

Subscripts and Superscripts

α	= α-level cut
$\underline{(\cdot)}$, $\overline{(\cdot)}$	= lower and upper bound/limit

Introduction

WITH the development of integrated design and manufacturing, many engineering systems tend to be more intractable for an analytical solution in closed form. Such a situation is encountered, for example, in the case of concurrent engineering due to the increased complexity of the integrated system(s). Under this circumstance, numerical techniques such as the finite element method (FEM) appear more useful in finding an approximate solution. The FEM has become one of the most popular and powerful tools for the analysis and design of engineering systems due to its general applicability and flexibility.

The conventional FEM for analysis and design of engineering systems neglects uncertainties present in an engineering process and assumes that the system information is known in precise terms. As such, the resulting solution is a single point in the parameter space. The effect of design and/or manufacturing uncertainties on the solution is studied by conducting a sensitivity analysis. The solution will be enlarged from a single point to an expanded region when the uncertainties of the parameters are considered in the analysis. The enlarged solution region can be used to satisfy specific requirements during the design process.

Unfortunately, most finite element software packages currently being used are more suitable at an advanced (or detail) design stage, when a designer has developed sufficient confidence in the design scheme. Furthermore, such software packages also have many embedded assumptions, for example, geometric dimensions need to be known exactly. Thus, a gap exists between the use of advanced software and the early (or conceptual/preliminary) stage of engineering analysis and design. The motivation of this work is to bridge this gap by developing a new procedure to incorporate various uncertainties into the current FEM.

The uncertainties associated with an engineering system can be grouped into three categories, subjective, objective, and hybrid type uncertainties. The subjective type of uncertainty is usually known as fuzziness and arises from cognitive sources involving, for instance, definition of certain parameters, human factors, and definition of the interrelationships among the parameters of the problems. The objective type of uncertainty, often referred to as randomness, results from noncognitive sources involving, for instance, physical stochastic likelihood, as well as statistical uncertainty due to incomplete information collected. The hybrid type of uncertainty is an integration of subjective and objective uncertainties due to the presence of both cognitive and noncognitive sources involving, for instance, simplifying assumptions in analytical and prediction models, simplified methods, idealized representations of real performances, and empirical data or formulation.[1]

The FEM in the presence of stochastic uncertainties has been well developed and extensively studied.[2–7] The stochastic FEM is used to handle uncertain (input) parameters described by probability distributions. The technique was initiated in the 1980s to account for

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stochastic uncertainties in system parameters (pertaining to material properties), geometry (pertaining to manufacturing errors, measurement, or instrumentation limitations), and external actions (pertaining to occasional disturbances from outside or environment). In 1980, a generic stochastic FEM was developed by Contreras[2] for modeling and analyzing structures in a probabilistic framework. The transient structural loads, idealized as stochastic processes, were incorporated into the finite element dynamic models with uncertain parameters. Handa and Anderson[3] proposed an FEM that allows estimates for the mean values, standard deviations, and correlation coefficients of structural displacements and stress by considering variations in applied loads, dimensions, and material properties. Nakagiri et al.[4] developed a method for the uncertain eigenvalue analysis of fiber-reinforced plastic plates. By treating the fiber orientations and thickness of plies as random variables, the coefficients of variation of the eigenfrequency were computed. Vanmarcke and Grigoriu[5] presented a stochastic FEM for solving a variety of engineering mechanics problems in which physical properties exhibit one-dimensional spatial random variation. The basic concepts underlying random loads and material properties involved in structural engineering were examined, as well as the various stochastic finite element formulations that cover a wide range of applications to both static and dynamic analyses.[6] Kleiber and Hien[7] presented the developmental steps from basic probability theory to stochastic FEM.

The FEM in the presence of fuzzy uncertainties has been investigated recently by several researchers.[8–11] The fuzzy FEM is used to manipulate uncertain (input) parameters described by possibility distributions. The technique was developed beginning in the early 1990s to capture fuzzy uncertainties associated with subjective preference or personal desire, during a design development period, in imprecisely describing, selecting, or estimating data/formulation for system parameters, geometry, and external actions. Rao and Sawyer[8] proposed a fuzzy FEM for static analysis of engineering systems using an optimization-based scheme for the numerical solution of systems of fuzzy linear equations. The fuzzy treatment of system parameters, geometry, and applied loads was considered and implemented in their approach. Chen and Rao[9] developed a fuzzy FEM for vibration analysis of imprecisely defined systems by using a search-based algorithm. Their approach enhances the computational efficiency in fuzzy operations for identifying the system dynamic responses. Valliappan and Pham[10] considered the use of fuzzy finite element analysis for a foundation on an elastic soil medium. In their work, the elastic modulus and Poisson's ratio of the soil were taken as fuzzy parameters to represent the uncertainty in the soil behavior. Shimuzu and Hiroaki[11] used fuzzy sets as a basis to automatically generate the finite element mesh. The fuzzy set theory was utilized in their method to mathematically model the human thought process.

The FEM in the presence of hybrid uncertainties has not been considered in the literature. This work will address and discuss the application of FEM to engineering systems under hybrid uncertainties. This implies that the effects of both input randomness and fuzziness will be taken into account simultaneously in determining the system responses. In the following sections, the basic stochastic and fuzzy FEMs are reviewed first. Then, the FEM in the presence of hybrid uncertainties is presented and discussed. Next, the proposed method is illustrated by two numerical examples. Finally, the present work is overviewed with concluding remarks.

Background

Depending on the nature and extent of uncertainty, the randomness and fuzziness present in an engineering system can be distinguished as follows. If only the frequencies of occurrence of system/design parameters are to be captured with known characteristics, the performance or output of the system can be determined using the theory of probability. A probabilistic or stochastic problem is characterized by the probability distribution associated with the uncertain parameters. In this case, the probability quantifies the measure of the frequency of occurrence of an event. Fuzzy theory, on the other hand, can be used to predict the pertinent system response if the system/design parameters are expressed in linguistic or imprecise terms. A fuzzy problem is characterized by the possibility distribution defined by an appropriate membership function. In this case, the possibility quantifies the imprecision or meaning of an event and measures the extent to which a sample is close to a desired element of a population. The basic probability and fuzzy theories can be found in Refs. 12–14. The stochastic or fuzzy FEM essentially follows the basic framework of the traditional (deterministic) FEM except for the procedure in solving the governing equations. The basic steps involved are essentially the same as those in the deterministic finite element analysis.[15]

Consider the following system of uncertain [uncertain (uncertainty) is used to imply either stochastic (randomness) or fuzzy (fuzziness)] linear equations:

$$H(\boldsymbol{A}(\boldsymbol{P}), \boldsymbol{X}) = \boldsymbol{B}(\boldsymbol{P}) \tag{1}$$

where

$$H(\boldsymbol{A}, \boldsymbol{X}) = \boldsymbol{A}\boldsymbol{X} \tag{2}$$

The objective is to find the uncertain vector $\boldsymbol{X} = (x_1, x_2, \ldots, x_n)^T$ that satisfies Eqs. (1) and (2), where $\boldsymbol{A} = (a_{ij})_{n\times n}$ $(i, j = 1, 2, \ldots, n)$ and $\boldsymbol{B} = (b_1, b_2, \ldots, b_n)^T$ denote the input uncertain coefficient matrix and uncertain right-hand-side vector, respectively. Note that for most engineering systems, the coefficient matrix $\boldsymbol{A}$ and the constant vector $\boldsymbol{B}$ may be dependent on the vector of the input parameters $\boldsymbol{P}$. The methods of solving a system of finite element equations in the presence of each type of uncertainty are outlined next.

Stochastic FEM

The stochastic FEMs are used to predict the statistical uncertainty likely to be present in the results of an engineering model due to random uncertainties in the model-input parameters. It is assumed that the stochastic data of all input variables are known a priori. If the joint probability density function of the random input parameters is known, then the mean and variance of a function of the random variables, $g(p_1, p_2, \ldots, p_n)$, can be determined using the principles of probability theory.

On the other hand, if the joint probability density function is not known, then the mean and variance of $g(\cdot)$ can be approximated by expanding the function, $g(p_1, p_2, \ldots, p_n)$, in a Taylor's series about the mean values of all input variables. If the Taylor's series is truncated after the second-order terms, then $g(p_1, p_2, \ldots, p_n)$ is approximated by

$$g(p_1, p_2, \ldots, p_n) \approx g(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial p_i}\right|_{(\mu_1, \mu_2, \ldots, \mu_n)} (p_i - \mu_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left.\frac{\partial^2 g}{\partial p_i \partial p_j}\right|_{(\mu_1, \mu_2, \ldots, \mu_n)} (p_i - \mu_i)(p_j - \mu_j) \tag{3}$$

This yields the mean and variance of $g(\cdot)$ as

$$\mu(\boldsymbol{Y}) = g(\mu_1, \mu_2, \ldots, \mu_n) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left.\frac{\partial^2 g}{\partial p_i \partial p_j}\right|_{(\mu_1, \mu_2, \ldots, \mu_n)} \mathrm{cov}[p_i, p_j] \tag{4}$$

$$\sigma^2(\boldsymbol{Y}) = \sum_{i=1}^{n} \left(\left.\frac{\partial g}{\partial p_i}\right|_{(\mu_1, \mu_2, \ldots, \mu_n)} \right)^2 \mathrm{var}[p_i] + \sum_{i=1}^{n}\sum_{j=1}^{n} \left.\frac{\partial^2 g}{\partial p_i \partial p_j}\right|_{(\mu_1, \mu_2, \ldots, \mu_n)} \mathrm{cov}[p_i, p_j] \tag{5}$$

which are known as the partial derivative rule.[12] The partial derivative rule can be used in conjunction with the FEM by treating the finite element solution found by solving the system in Eq. (1) as a function of random variables. Using the approximations shown in Eqs. (4) and (5), output statistics for the finite element problem can be generated. If the gradients are not explicitly known, a finite

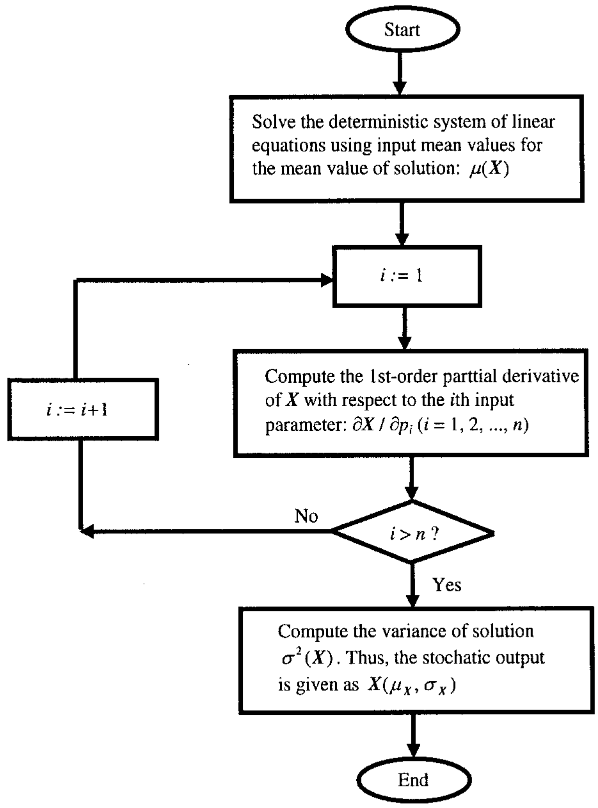


Fig. 1 Solution procedure for a system of stochastic linear equations.

difference approximation can be used. This is known as a perturbation approach to stochastic FEM. The procedure to be followed in finding the output statistics of a stochastic linear system is shown as a flow diagram in Fig. 1.

Fuzzy FEM

Set theory usually defines a set such that a given entity must either fully belong to a set or not belong to the set. A fuzzy set, on the other hand, is defined with the use of a membership function $m_i(\cdot) \in [0, 1]$ where $m_i(\cdot) = 1$ means total membership in the fuzzy set, $m_i(\cdot) = 0$ means no membership in the fuzzy set, and fractional values represent partial membership in the fuzzy set. The term α level or α cut refers to a discretization of a fuzzy number or membership function by giving the lower and upper bounds that indicate a given membership value of α. In other words, this indicates the range of values that belong to a given fuzzy set with a membership value of α or greater.

As stated earlier, the fuzzy FEM follows exactly the same routine as used in the deterministic FEM except for the solution of a system of linear equations. Several remarkable works have been reported[8,9,16] for solving a system of fuzzy linear equations. The computational methodology used for the numerical solution of a set of fuzzy linear equations involves three steps: 1) computerized selection of fuzziness (to ensure existence of the solution), 2) implementation of fuzzy operations (to save intermediate computational effort), and 3) execution of a search-based algorithm (to quickly identify the solution). The method basically transforms the original problem of solving a fuzzy linear system into an equivalent problem of solving a system of nested interval linear equations by using interval analysis. Thus, the fuzzy linear system, defined by Eqs. (1) and (2), can be expressed alternatively in the following interval form at a specific α level:

$$H_\alpha(\boldsymbol{A}_\alpha,\ \boldsymbol{X}_\alpha) = \boldsymbol{B}_\alpha \tag{6}$$

with

$$H_\alpha(\boldsymbol{A}_\alpha,\ \boldsymbol{X}_\alpha) = \boldsymbol{A}_\alpha \boldsymbol{X}_\alpha \tag{7}$$

where $\boldsymbol{A}_\alpha = (a_{ij,\alpha})_{n\times n}$ $(i, j = 1, 2, \ldots, n)$, $a_{ij,\alpha} = [\underline{a}_{ij}, \bar{a}_{ij}]_\alpha$; $\boldsymbol{B}_\alpha = (b_{i,\alpha})_{n\times 1}$, $b_{i,\alpha} = [\underline{b}_i, \bar{b}_i]_\alpha$; and $\boldsymbol{X}_\alpha = (x_{1,\alpha}, \ldots, x_{i,\alpha}, \ldots, x_{n,\alpha})^T$, $x_{i,\alpha} = [\underline{x}_i, \bar{x}_i]_\alpha$. Note that the subscript α denotes that the variable or function is considered at a specified α level and that a bar below or above a variable or function represents a lower or upper bound, respectively, of the variable or function.

In the search-based numerical method, the lower and upper bounds of the constant term $\underline{b}_i$ and $\bar{b}_i$ are taken as the two target values for the ith search with respect to the ith fuzzy linear equation at a specific α level, namely, the ith interval linear equation $h_i(A_{i,\alpha}, \boldsymbol{X}_\alpha) = b_{i,\alpha}$, where $h_i(A_{i,\alpha}, \boldsymbol{X}_\alpha) = h_i[(a_{i1}, a_{i2}, \ldots, a_{in})_\alpha, \boldsymbol{X}_\alpha]$ for $i = 1, 2, \ldots, n$. Each interval solution is sought by means of two separate searches, one directed toward the lower bound of the target value and the other toward the upper bound of the target value. The conditions under which a solution exists for the fuzzy system can be stated in the form of inequalities, at any prescribed level α, as

$$\underline{b}_i < h_i\left(A_{i,\alpha}, \boldsymbol{X}^{(0)}\right) < \bar{b}_i, \qquad i = 1, 2, \ldots, n \tag{8}$$

or

$$\underline{\boldsymbol{B}}_\alpha < H_\alpha\left(\boldsymbol{A}_\alpha, \boldsymbol{X}^{(0)}\right) < \bar{\boldsymbol{B}}_\alpha \tag{9}$$

where $\boldsymbol{X}^{(0)}$ indicates the crisp solution (vector) found from the crisp system of linear equations (without any consideration of fuzziness).

The search algorithm starts with the solution of the crisp system of linear equations. The resulting crisp solution $\boldsymbol{X}^{(0)}$ is used as the base or central solution to find the fuzzy solution. The subsequent search process for the solution, at a specific α level, is deployed on the basis of the crisp solution. Thus, the search objective for each equation can be identified by targeting two vectors that indicate the left- and right-side variations, Δ_i^L (≥ 0) and Δ_i^R (≥ 0),

$$\underline{\boldsymbol{X}}_i = \boldsymbol{X}^{(0)} - \Delta_i^L, \qquad \bar{\boldsymbol{X}}_i = \boldsymbol{X}^{(0)} + \Delta_i^R \tag{10}$$

where $\Delta_i^L = (\delta_1^L, \delta_2^L, \ldots, \delta_m^L)_i^T$ and $\Delta_i^R = (\delta_1^R, \delta_2^R, \ldots, \delta_m^R)_i^T$. Once all local solutions are found from the various interval equations, the global solution at a specific α level, $\boldsymbol{X}_\alpha$, is determined as the intersection of the individual local solutions, $\boldsymbol{X}_{i,\alpha}$ $(i = 1, 2, \ldots, n)$. Thus, the solution of the system of fuzzy linear equations at any specific α level can be formalized as

$$\boldsymbol{X}_\alpha = \boldsymbol{X}_{1,\alpha} \cap \boldsymbol{X}_{2,\alpha} \cap \cdots \cap \boldsymbol{X}_{n,\alpha} \tag{11}$$

where $\boldsymbol{X}_{i,\alpha} = [\underline{\boldsymbol{X}}_i, \bar{\boldsymbol{X}}_i]_\alpha$ for $i = 1, 2, \ldots, n$. The solution procedure is shown as a flowchart in Fig. 2.

Methodology

The unified FEM proposed closely parallels the stochastic/fuzzy finite element analysis procedure with the exception of the way to generate the final solution. A procedure is outlined in the following section to generate the solution space that allows the designer more freedom during the early or intermediate design period to pick an appropriate design in the expanded solution space. The proposed unified FEM is a tool to solve for the (uncertain) system response parameters given both the stochastic and fuzzy information of the input parameters.

Unified Solution Scheme

In a design engineering process, a remarkable difference exists among the three types of uncertainty. In general, a designer cannot control the objective aspect of uncertainty inherent in, for example, the material properties and system performance or the qualities due to a lack of exact or complete knowledge. However, designers can have their subjective preferences or desires in choosing/specifying values for design parameters to be used in the formulation of the system. Typically, the hybrid uncertainty interacts with each phase of any engineering design development. At the early design stage, the subjective uncertainty, representing the design imprecision or degree of vagueness/inexactness in choosing among design alternatives,[17] usually dominates the preliminary design configuration. With the progress of the iterative design process, this type of uncertainty is reduced gradually and will be eliminated in the final design. The objective uncertainty, on the other hand, usually remains (by retaining a certain level) throughout the design process. Comprehensively,

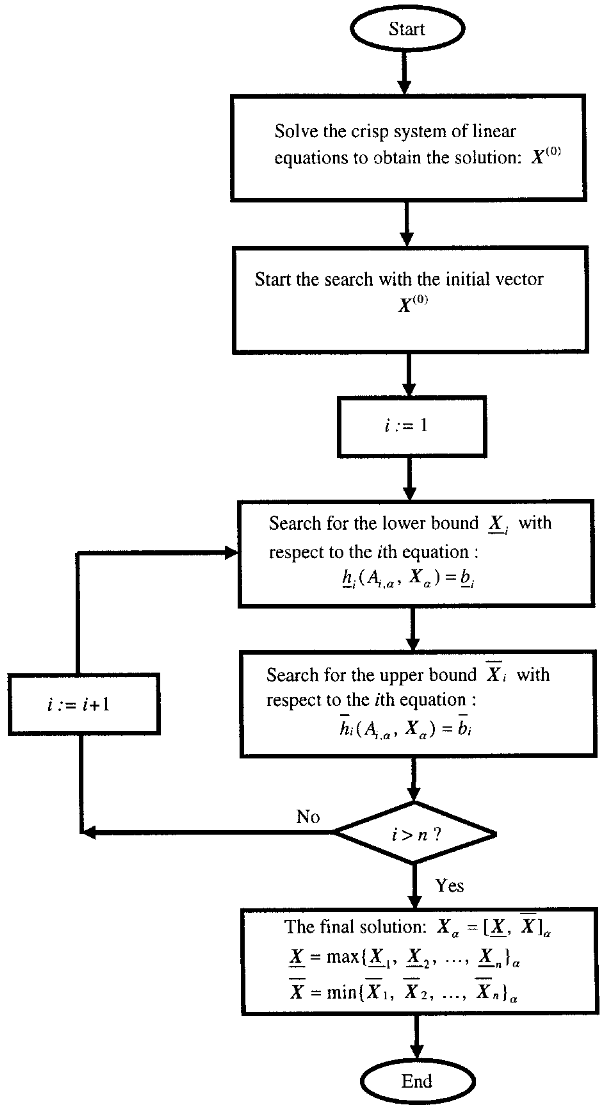


Fig. 2 Solution procedure for a system of fuzzy linear equations.

any engineering design process can be viewed as an evolutionary development procedure in which the overall uncertainty is shrunk globally with the progression of the iterative design process.

If the FEM is used at an early design stage, two distinct forms of uncertainty exist simultaneously: stochastic uncertainty and design imprecision. The stochastic uncertainty associated with the geometry or material properties of the structure or the loads applied may be represented as random data, particularly if the values are measured in an experiment. This form of uncertainty is spatially distributed over the region of the structure and is modeled as a stochastic or random field. On the other hand, design imprecision is usually involved in subjective selection of the parameter values among design alternatives subjected to personal desirability or knowledge. The fuzzy uncertainty conveys imprecise information present in description/selection of the geometry, material properties, applied loads, or boundary conditions of a system. As a result, all of the input parameters may be treated generally as nothing but hybrid-form uncertain parameters. This leads to the need to upgrade the existing FEM by integrating the stochastic and fuzzy uncertainties together in the contextual solution procedure. Therefore, in the proposed unified FEM, the random and fuzzy uncertainties are considered simultaneously with each parameter. For conciseness, the input parameters are distinguished by the two categories, empirical parameters (indicating material properties and external actions) and design parameters (indicating geometric dimensions), which can be formalized as follows:

$$\boldsymbol{P} = (\boldsymbol{P}_e, \boldsymbol{P}_d)^T \tag{12}$$

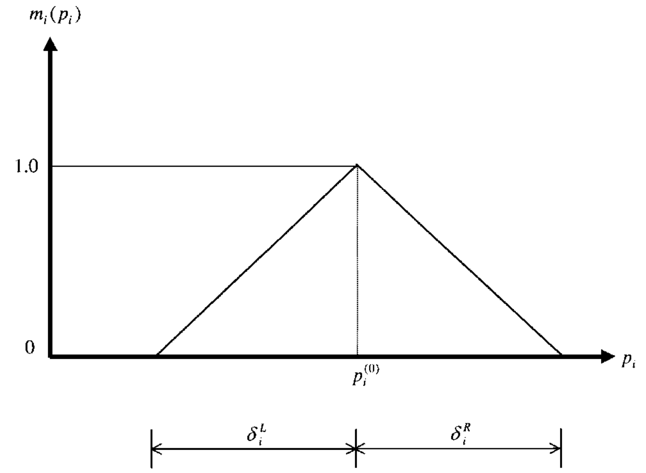


Fig. 3 Triangular membership function of the ith input parameter.

with

$$\boldsymbol{P}_e \subset \boldsymbol{P}, \qquad \boldsymbol{P}_d \subset \boldsymbol{P}, \qquad \boldsymbol{P}_e \cap \boldsymbol{P}_d = \emptyset \tag{13}$$

and

$$\begin{aligned} \boldsymbol{P}_e &= (e_i)_{k\times 1} = (e_1, e_2, \ldots, e_k)^T \\ \boldsymbol{P}_d &= (d_i)_{l\times 1} = (d_1, d_2, \ldots, d_l)^T \\ \boldsymbol{P} &= (p_i)_{(k+l)\times 1} = (p_1, p_2, \ldots, p_{k+l})^T \end{aligned} \tag{14}$$

where $\boldsymbol{P}$, $\boldsymbol{P}_e$, and $\boldsymbol{P}_d$ represent the vectors of input, empirical, and design parameters, respectively. In general, the empirical parameters in $\boldsymbol{P}_e$ have stochastic data, and the designer must assign a suitable fuzzy description. Meanwhile, the designer usually has a predetermined fuzzy description of the design parameters in $\boldsymbol{P}_d$, and an appropriate stochastic definition must be generated.

In this work, triangular type membership functions, i.e., possibility distributions, are considered, for simplicity, to model fuzziness (or design imprecision) associated with all the input parameters. (Note that, in the following sections, the term input parameters refers to both empirical and design parameters.) The L–R form representation proposed by Dubois and Prade[18] is used (shown in Fig. 3), which can be defined as

$$m_i(p_i) = \begin{cases} 1 + \dfrac{p_i - p_i^{(0)}}{\delta_i^L}, & p_i^{(0)} - \delta_i^L \le p_i < p_i^{(0)} \\ 1 - \dfrac{p_i - p_i^{(0)}}{\delta_i^R}, & p_i^{(0)} \le p_i < p_i^{(0)} + \delta_i^R \\ 0, & \text{otherwise} \end{cases}$$

$$(i = 1, 2, \ldots, k + l) \tag{15}$$

where $m_i(\cdot)$ is the membership function of the ith input parameter, δ_i^L (>0) and δ_i^R (>0) are the left and right spreads that are termed as fuzziness, and $p_i^{(0)}$ is the mode, i.e., nominal value, whose membership function corresponds to the value of 1, as shown in Fig. 3. Alternatively, the ith fuzzy input parameter is simply denoted as $(p_i^{(0)}, \delta_i^L, \delta_i^R)$ for $i = 1, 2, \ldots, k + l$.

In most real-life situations, an empirical parameter such as Young's modulus is derived from an experiment. The experimentally collected or measured data usually follow normal distribution due to the stochastic nature of any experiment. As an example, the Young's modulus at any one point in the body of material may differ slightly from any other point. To quantify the stochastic uncertainty associated with an empirical parameter, normal distribution is considered in this work with the following density function for the ith empirical parameter:

$$f_i(e_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\left(\frac{e_i - \mu_i}{\sigma_i}\right)^2\right] \qquad (i = 1, 2, \ldots, k) \tag{16}$$

where μ_i and σ_i are the mean value and standard deviation of the ith empirical parameter. During the early design period, an arbitrary value needs to be specified for an empirical parameter as an input to the system. Because of the fuzzy nature of the early design stage, this specification is totally subjective, depending on the designer's preference. Mostly, a mean value of experimental data is used as the nominal value of the system parameter in the contextual formulation when no other information is available. If additional information is provided on functional requirements, such as reliability or safety issues involved, then the nominal value selected may deviate slightly from its mean value but still fall in a possible range, i.e., fuzziness, which is quantified as the sum of left and right spreads.

On the other hand, a design parameter such as the length of a beam represents a parameter derived from a design process. In the case of integrated design and manufacturing, manufacturing disturbances (or errors) need to be incorporated into the design iterations in which each design parameter is coupled with an associated range indicating a manufacturing tolerance present in a specific physical process. Typically, the range representing the manufacturing tolerance, for example, Δd_i, can be treated similarly to the range of fuzziness, i.e., $\delta_i^L + \delta_i^R$,

$$\Delta d_i = \delta_i^L + \delta_i^R \qquad (i = 1, 2, \ldots, l) \quad (17)$$

in which the nominal value can vary along with a satisfaction level indicated by an appropriate membership function value. Because of the imprecise nature of design parameters in the early design stage, any possible numbers within this tolerance range can be evenly selected as potential candidates for nominal values of the design parameters. Thus, the stochastic aspect of a specific manufacturing process associated with a design parameter may be formulated in the form of a uniform distribution

$$f_i(d_i) = \begin{cases} \dfrac{1}{\delta_i^L + \delta_i^R}, & d_i^{(0)} - \delta_i^L \le d_i < d_i^{(0)} + \delta_i^R \\ 0, & \text{otherwise} \end{cases} \qquad (i = 1, 2, \ldots, l) \quad (18)$$

where $d_i^{(0)}$ is the nominal value of the ith design parameter. The mean and standard deviation of uniform distribution can be identified as

$$\mu_i(d_i) = d_i^{(0)} + \frac{\delta_i^R - \delta_i^L}{2} \qquad (19)$$

and

$$\sigma_i(d_i) = \frac{\delta_i^L + \delta_i^R}{\sqrt{3}} \qquad (20)$$

with $\delta_i^L + \delta_i^R = \Delta d_i$, as indicated in Eq. (17). Thus, an approximate interrelation between the stochastic and fuzzy uncertainties present in the early design period is established for the design parameters. Given either the stochastic or the fuzzy information, the other one can be derived and inferred readily using Eqs. (17–20). For simplicity, the uniform variate is approximated as an equivalent normal variate with standard deviation equal to $\Delta d_i/\sqrt{3}$. The functional mean and standard deviation values involving the empirical and design (or input) parameters can be computed using the partial derivative rule.[12] The subsequent procedure for solving the system of finite element equations under hybrid uncertainties involves three major steps, as discussed next. Figure 4 shows a schematic of the unified solution procedure. The added computational time involved in obtaining the unified solution of a hybrid problem is minimal compared to the computational effort associated with the stochastic and fuzzy finite element solutions.

Step 1: Solution of Stochastic Finite Element Problem

In the first stage, the stochastic finite element problem is solved to identify the probability distribution(s) of the system (output) response, for example, $f(X)$, in the absence of fuzzy type uncertainty. The stochastic FEM problem is solved using the following procedure. 1) Solve for the mean value of the FEM output using mean values for all input parameters, as in Eq. (4). 2) Compute the first derivatives of the output with respect to all input parameters. 3) Compute the output variance according to Eq. (5).

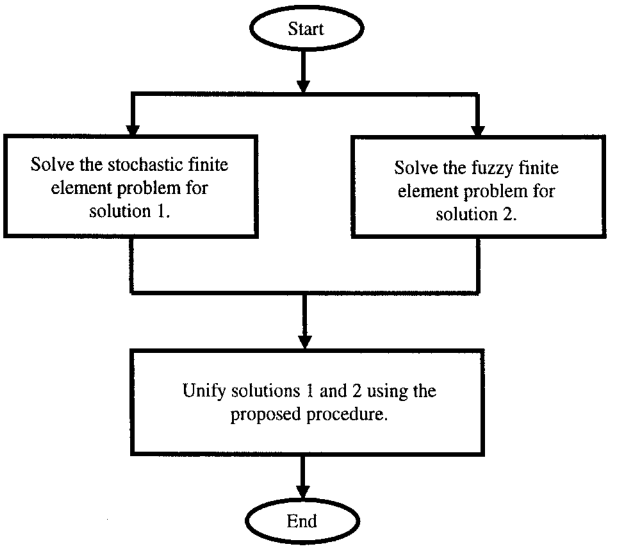


Fig. 4 Unified solution procedure flow.

Step 2: Solution of Fuzzy Finite Element Problem

In the second stage, the fuzzy finite element problem is solved to identify the possibility distribution(s) of the system (output) response, for example, $m(X)$, in the absence of stochastic type uncertainty. A search-based algorithm, involving the following step-by-step procedure, is used. 1) Solve the deterministic finite element problem for the crisp solution by setting all of the input parameters equal to their respective nominal values. 2) Detect the search directions for both (left and right) sides of the unknowns using probe lengths. 3) Select the optimum settings of components of the unknowns using a Taguchi-based approach. 4) Deploy the search with accelerated steps until the region in which the solution will lie is identified. 5) Apply the bisection technique to accelerate convergence to a local solution. 6) Repeat the preceding steps for all of the linear equations to find all of the local solutions. 7) Determine a global solution using the intersection of all local solutions, as defined by Eq. (6).

Step 3: Unified Solution of the Hybrid Problem

In the last stage, a unified solution is generated for the finite element problem under hybrid uncertainties by combining the solutions, $f(X)$ and $m(X)$, found earlier. Because the stochastic and fuzzy uncertainties are simultaneously present in the system a measure known as fuzzy probability[19] can be used to determine the hybrid-uncertainty mean (m_X) and variance (ν_X), as follows:

$$m_X = \frac{\int_\Omega X \cdot m(X) f(X)\,\mathrm{d}X}{\int_\Omega m(X) f(X)\,\mathrm{d}X} \qquad (21)$$

and

$$\nu_X = \frac{\int_\Omega (X - m_X)^2 \cdot m(X) f(X)\,\mathrm{d}X}{\int_\Omega m(X) f(X)\,\mathrm{d}X} \qquad (22)$$

where Ω is the universe of discourse to which X belongs, $m(X)$ is the piecewise linear membership function obtained as the output of the fuzzy FEM solution, and $f(X)$ is the probability density function [Eq. (16)] using the output mean and standard deviation obtained from the stochastic FEM solution. In numerical implementation, Eqs. (21) and (22) are transformed into an equivalent form in which a nested set of appropriate interval functions are to be evaluated at specified α-cut levels, as discussed next.

Hybrid Uncertainty Measures

The hybrid-uncertainty mean and variance are measures that are useful to a designer in the initial stages of the design phase. Their purpose is to combine the information generated by both stochastic and fuzzy finite element calculations already performed. The hybrid-uncertainty mean is useful as a best guess for the representative output expected for the preliminary design with a given level of input fuzziness (denoted by the α cut) and known random inputs. The hybrid-uncertainty variance (or the hybrid-uncertainty spread defined as the square root of the hybrid-uncertainty variance) is a measure of the size of the predicted output space both in terms of fuzziness and probability. A small hybrid-uncertainty variance at a given α level indicates that the output solution is predicted with a relatively small degree of uncertainty.

The determination of the hybrid-uncertainty mean and variance requires the evaluation of three integrals. [Note that the denominators of Eqs. (21) and (22) are the same.] For a given output variable at a specified α cut, the universe of discourse Ω corresponds to the range of the fuzzy solution given by $x_{i,\alpha} = [\underline{x}_i, \bar{x}_i]_\alpha$. The limits of integration are, therefore, $\underline{\boldsymbol{X}}_\alpha$ and $\bar{\boldsymbol{X}}_\alpha$. If $\alpha \neq 1$, Eqs. (21) and (22) can be solved using an appropriate numerical integration technique such as an adaptive quadrature method. [Note that the membership function $m(\boldsymbol{X})$ is a piecewise-linear function and that the probability density function $f(\boldsymbol{X})$ is a smooth Gaussian distribution.]

For $\alpha = 1$, the upper and lower limits of integration will be identical, resulting in an indeterminate condition, i.e., $0 \div 0$. Application of L'Hôpital's rule gives

$$m_X = \lim_{X \to X^{(0)}} \frac{(\mathrm{d}/\mathrm{d}\boldsymbol{X}) \int_\Omega \boldsymbol{X} \cdot m(\boldsymbol{X}) f(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}}{(\mathrm{d}/\mathrm{d}\boldsymbol{X}) \int_\Omega m(\boldsymbol{X}) f(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}}$$

$$= \left.\frac{\boldsymbol{X} \cdot m(\boldsymbol{X}) f(\boldsymbol{X})}{m(\boldsymbol{X}) f(\boldsymbol{X})}\right|_{X^{(0)}} = \boldsymbol{X}^{(0)} \tag{23}$$

and

$$v_X = \lim_{X \to X^{(0)}} \frac{(\mathrm{d}/\mathrm{d}\boldsymbol{X}) \int_\Omega (\boldsymbol{X} - m_X)^2 \cdot m(\boldsymbol{X}) f(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}}{(\mathrm{d}/\mathrm{d}\boldsymbol{X}) \int_\Omega m(\boldsymbol{X}) f(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}}$$

$$= \left.\frac{\left(\boldsymbol{X} - \boldsymbol{X}^{(0)}\right)^2 \cdot m(\boldsymbol{X}) f(\boldsymbol{X})}{m(\boldsymbol{X}) f(\boldsymbol{X})}\right|_{X^{(0)}} = \boldsymbol{0} \tag{24}$$

This shows that the hybrid-uncertainty mean is identical with the fuzzy mode, and the hybrid-uncertainty variance is zero at an α level of 1.0.

Because probability theory can be viewed as a subset of possibility theory,[19] the hybrid-uncertainty results are to be interpreted more as fuzzy information than stochastic information. In a sense, the hybrid-uncertainty mean and variance use the known stochastic information to modify the fuzzy results. The hybrid-uncertainty mean can be geometrically interpreted as a weighted centroid calculation. It is typical to interpret fuzzy results through some form of defuzzification. A well-known defuzzification method involves the calculation of the centroid of a fuzzy result. This hybrid-uncertainty mean uses the probability density function obtained from a stochastic FEM analysis as a weighting factor. The result is a best guess for the design output both in terms of fuzziness (it is near the midpoint of the fuzzy range) and in terms of probability (it is shifted toward the most probable solution).

The hybrid-uncertainty spread should not be confused with the probabilistic standard deviation. It cannot be assumed that a given fraction of the results will reside within a range generated by the hybrid-uncertainty mean plus or minus some multiple of the hybrid-uncertainty spread. Rather, the hybrid-uncertainty spread can be used to compare the relative confidence with which a given output is known for different α levels.

Examples

Two examples are presented to illustrate the computational aspects of the proposed method. In each example, the input parameters are assumed to contain both stochastic and fuzzy uncertainties. The stochastic inputs are considered to be independent with no covariance following normal distributions, and the fuzzy parameters are characterized by triangular type fuzzy quantities.

Three-Stepped Bar

The first example deals with the displacement analysis of the three-stepped bar, shown in Fig. 5. In this example, all input parameters are considered to be both fuzzy and stochastic in nature. The cross-sectional areas (A_i, $i = 1, 2, 3$) and lengths (L_i, $i = 1, 2, 3$) are considered to be the design parameters following uniform distribution. All of the empirical parameters, including Young's modulus (E_i, $i = 1, 2, 3$) and external loads (F_i, $i = 1, 2, 3$), are assumed to follow normal distribution. The fuzzy inputs are shown in Table 1, and the stochastic inputs are summarized in Table 2. (Data for uncertain input parameters for the second example are given in Table 3.)

Table 1 Data for the fuzzy aspect of input parameters: three-stepped bar

Parameter	(Mode, L spread, R spread)	Parameter	(Mode, L spread, R spread)
A_1, in.^{2}	(3.00, 0.10, 0.20)	E_1, psi	(3.0e7, 2.0e6, 3.0e6)
A_2, in.^{2}	(2.00, 0.20, 0.10)	E_2, psi	(3.0e7, 2.0e6, 3.0e6)
A_3, in.^{2}	(1.00, 0.10, 0.10)	E_3, psi	(3.0e7, 2.0e6, 3.0e6)
L_1, in.	(12.00, 0.50, 0.30)	F_1, lb	(0.00, 1.5e3, 1.0e3)
L_2, in.	(10.00, 0.40, 0.30)	F_2, lb	(0.00, 6.0e3, 5.0e3)
L_3, in.	(6.00, 0.20, 0.20)	F_3, lb	(1.0e4, 7.9e3, 1.0e4)

Table 2 Data for the stochastic aspect of input parameters: three-stepped bar

Parameter	(Mean, standard deviation)	Parameter	(Mean, standard deviation)
A_1, in.^{2}	(3.05, 0.0866)	E_1, psi	(3.05e7, 1.4436e6)
A_2, in.^{2}	(1.95, 0.0866)	E_2, psi	(3.05e7, 1.4436e6)
A_3, in.^{2}	(1.00, 0.0577)	E_3, psi	(3.05e7, 1.4436e6)
L_1, in.	(11.9, 0.2309)	F_1, lb	(0.00, 1.0e-3)
L_2, in.	(9.95, 0.2021)	F_2, lb	(0.00, 1.0e-3)
L_3, in.	(6.00, 0.1155)	F_3, lb	(1.105e4, 5.167e3)

Table 3 Data for uncertain input parameters: two-dimensional heat conduction problem

Parameter	Fuzzy aspect (mode, L spread, R spread)	Stochastic aspect (mean, standard deviation)
k, W/cm-K	(30.0, 3.0, 4.0)	(30.5, 2.0207)
L, cm	(10.0, 2.0, 1.0)	(9.5, 0.866)
$\dot{q}_0$, W/cm^3	(100.0, 15.0, 10.0)	(97.5, 7.2169)
T_∞, °C	(50.0, 5.0, 6.0)	(50.5, 3.1754)

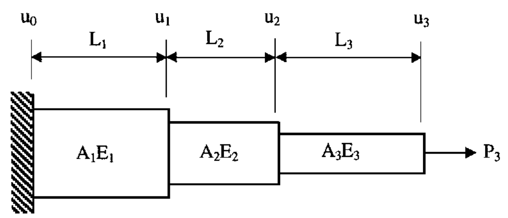


Fig. 5 Example 1: three-stepped bar problem.

The results of the stochastic and fuzzy analyses are shown in Table 4. The hybrid-uncertainty results for the first displacement component are given in Table 5. Figure 6 shows the hybrid uncertainty mean (dotted line) superimposed on the fuzzy results (solid line). The purpose of the hybrid-uncertainty mean is to serve as a representative best guess of the output solution with regard to both the stochastic and fuzzy inputs. This parameter represents the solution that has a high degree of possibility as well as a high degree of probability. The hybrid spread shown in Fig. 7 indicates the relative degree of uncertainty that a designer must accept for a given α level. In this problem, all solutions from $\alpha = 0.75$ to 1.0 have a small level of uncertainty compared to all lower α levels. This indicates that the designer is free to choose any input values within the $0.75 \le \alpha \le 1.0$ input range without changing the uncertainty in the output solution very much. Inputs corresponding to $\alpha \le 0.75$ result in an increasingly large output uncertainty.

Two-Dimensional Heat Conduction

The second example deals with the temperature distribution in a square region with uniform energy generation, as shown in Fig. 8. It

Table 4 Fuzzy and stochastic aspects of output response at selected node(s)

Example 1: three-stepped bar			Example 2: two-dimensional heat conduction problem		
α-Cut level (fuzzy aspect)	Node 1		α-Cut level (fuzzy aspect)	Node 19	
	Lower bound	Upper bound		Lower bound	Upper bound
0.00	1.28659e-3	1.33941e-3	0.00	72.4532	75.1539
0.25	1.28659e-3	1.31341e-3	0.25	72.7411	74.7881
0.50	1.29959e-3	1.31341e-3	0.50	73.0267	74.4056
0.75	1.30000e-3	1.30041e-3	0.75	73.3101	74.0065
1.00	1.30000e-3	1.30000e-3	1.00	73.5907	73.5907
Mean value					
	1.41354e-3 (stochastic aspect)			70.9180 (stochastic aspect)	
Standard deviation					
	6.66268e-4			5.30629	

Table 5 Hybrid aspect of output response at selected node(s)

Example 1: three-stepped bar			Example 2: two-dimensional heat conduction problem		
	Node 1			Node 19	
α-Cut level	Hybrid mean	Hybrid spread	α-Cut level	Hybrid mean	Hybrid spread
0.00	1.30561e-3	1.05813e-5	0.00	73.7129	0.551703
0.25	1.30166e-3	7.00992e-6	0.25	73.7087	0.493497
0.50	1.30603e-3	3.95454e-6	0.50	73.6898	0.363245
0.75	1.30020e-3	1.17954e-7	0.75	73.6514	0.193837
1.00	1.30000e-3	0.0	1.00	73.5907	0.0

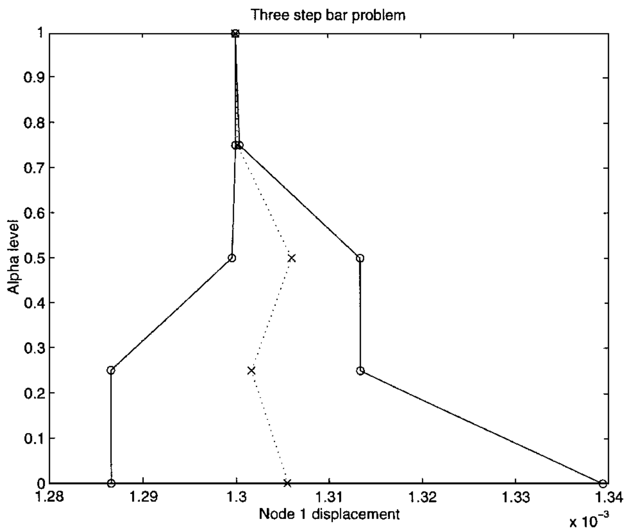


Fig. 6 Hybrid-uncertainty mean (· · · · ·) superimposed on fuzzy solution (——) for three-stepped bar example.

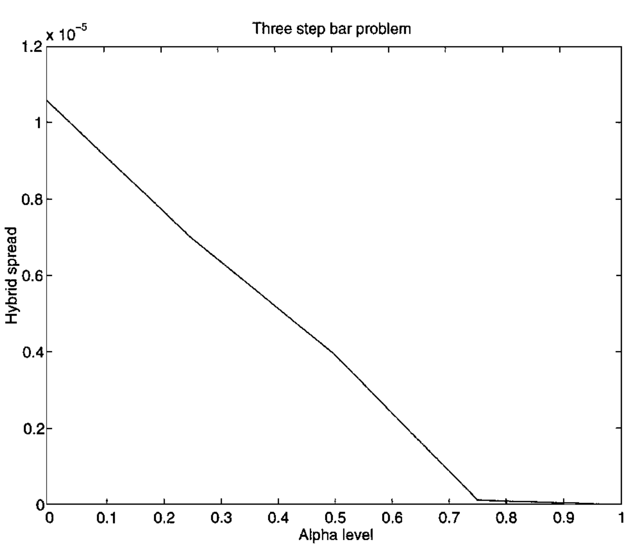
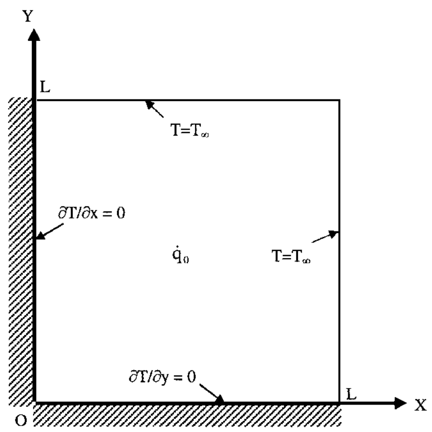
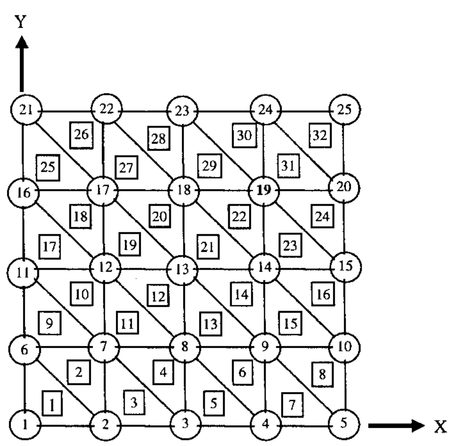


Fig. 7 Hybrid-uncertainty spread for three-stepped bar example.



Square region with uniform energy generation



Finite element idealization

Fig. 8 Example 2: two-dimensional heat conduction problem.

is assumed that there is no temperature variation in the z direction. Four system parameters are considered as inputs with one design parameter, side length L, and three empirical parameters, thermal conductivity k, heat flux $\dot{q}_0$, and surrounding temperature T_∞. The fuzzy and stochastic information for these inputs is given in Table 3. As shown in Fig. 8, the square region is modeled with 32 triangular type elements with 25 nodes; thus, there are 25 output values representing the nodal temperatures. Because the characteristics are

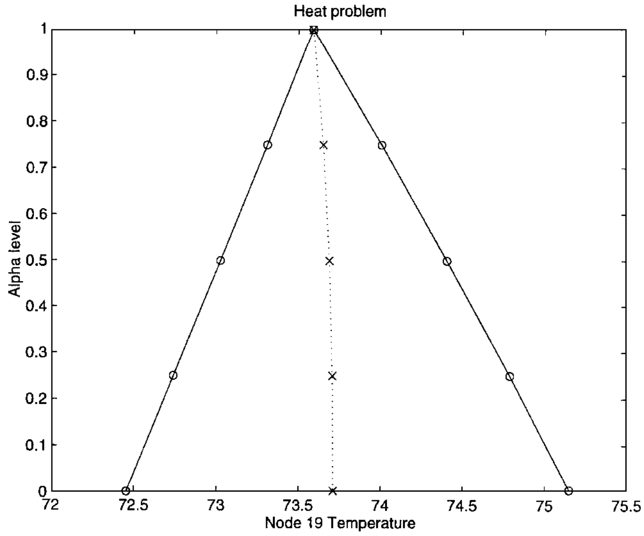


Fig. 9 Hybrid-uncertainty mean (· · · · ·) superimposed on fuzzy solution (——) for two-dimensional heat example.

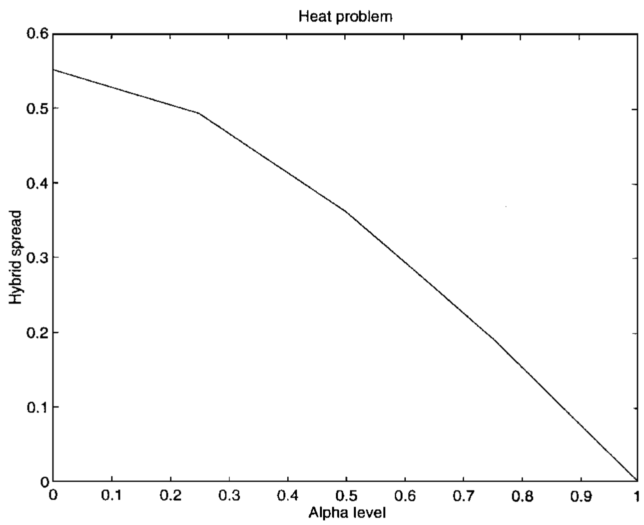


Fig. 10 Hybrid-uncertainty spread for two-dimensional heat example.

similar for all of the nodes, the results for node 19 are shown as a representative case.

The fuzzy and stochastic results for node 19 are shown in Table 4. The hybrid-uncertainty results are shown in Table 5. The hybrid-uncertainty mean of the heat conduction problem is shown superimposed on the fuzzy results in Fig. 9. The hybrid-uncertainty spread is shown in Fig. 10. Note, unlike the three-stepped bar case, the hybrid-uncertainty spread increases rapidly as the α level is reduced. This indicates that the selection of inputs with low α levels will result in a steadily increasing hybrid uncertainty. The designer must, therefore, choose an acceptable level of uncertainty and choose a corresponding α level that results in an uncertainty equal to or lower than this value.

Sensitivity Study

The stochastic data for the empirical input parameters are generally determined by experiments or some other objective phenomenon. The fuzzy information, on the other hand, is very much up to the discretion of the designer. For this reason, it is desirable to investigate the sensitivity of the hybrid results to a small change in the fuzzy results (presumably due to some minor adjustment of the fuzzy input parameters).

The hybrid-uncertainty results are generated using the same stochastic results, but perturbing the fuzzy results by 1.0% of their nominal values (at $\alpha = 1.0$). The new hybrid-uncertainty results are shown along with the original hybrid-uncertainty results in Fig. 11. Figure 11a shows that the hybrid-uncertainty mode (for a given α level) is shifted by 1.0% (same amount as the perturbation in the fuzzy data). This indicates that the hybrid-uncertainty mean is affected by a shift in the fuzzy results. On the other hand, Fig. 11b shows that there is little or no change in the hybrid-uncertainty spread with a perturbation in the fuzzy results. In other words, the relative level of uncertainty is unaffected by a slight change in the position of the hybrid-uncertainty mean.

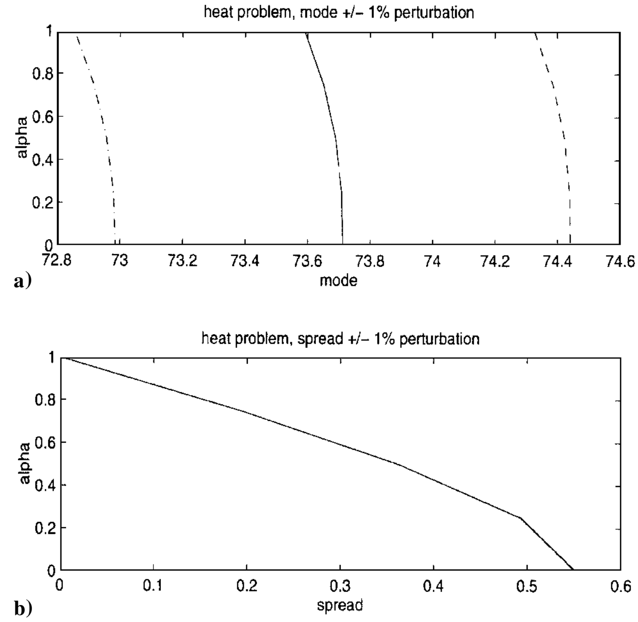


Fig. 11 Sensitivity study of two-dimensional heat conduction problem.

Conclusion

This work presents a procedure for the FEM in the presence of both fuzzy and stochastic uncertainties. The methodology generates fuzzy and stochastic finite element solutions and combines them in an appropriate manner to find the hybrid-uncertainty measures of hybrid-uncertainty mean and hybrid-uncertainty variance (or spread). These measures allow the designer to interpret FEM analysis at an early design phase.

The hybrid-uncertainty mean is useful as a best guess for FEM output in the presence of both fuzzy and stochastic inputs. This value represents a weighted defuzzification of the results where the stochastic probability distribution function acts as the weighting parameter. The hybrid-uncertainty variance or the hybrid-uncertainty spread is useful as a relative measure of the uncertainty of FEM solutions at different α levels. This allows designers to specify the allowable input parameter ranges by identifying the allowable uncertainty in the analysis output. The combination of fuzzy and stochastic information results in a solution that is feasible in both possibility and probability approaches. This level of uncertainty is nearly unaffected by slight changes in the hybrid-uncertainty mean.

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